Advances in Applied Mathematics **23**, 360–374 (1999)
Article ID aama.1999.0657, available online at http://www.idealibrary.com on IDEAL®

The Nullstellensatz for Systems of PDE[1]

Shiva Shankar[2]

Department of Electrical Engineering, Indian Institute of Technology, Powai, Bombay-400076, India

Received October 8, 1998; accepted December 15, 1998

Given an ideal $\mathbf{I}$ in $\mathscr{A}$, the polynomial ring in n-indeterminates, the affine variety of $\mathbf{I}$ is the set of common zeros in $\mathbb{C}^n$ of all the polynomials that belong to $\mathbf{I}$, and the Hilbert Nullstellensatz states that there is a bijective correspondence between these affine varieties and radical ideals of $\mathscr{A}$. If, on the other hand, one thinks of a polynomial as a (constant coefficient) partial differential operator, then instead of its zeros in $\mathbb{C}^n$, one can consider its zeros, i.e., its homogeneous solutions, in various function and distribution spaces. An advantage of this point of view is that one can then consider not only the zeros of ideals of $\mathscr{A}$, but also the zeros of submodules of free modules over $\mathscr{A}$ (i.e., of systems of PDEs). The question then arises as to what is the analogue here of the Hilbert Nullstellensatz. The answer clearly depends on the function–distribution space in which solutions of PDEs are being located, and this paper considers the case of the classical spaces. This question is related to the more general question of embedding a partial differential system in a (two-sided) complex with minimal homology. This paper also explains how these questions are related to some questions in control theory. © 1999 Academic Press

1. INTRODUCTION

Given an ideal $\mathbf{I}$ in $\mathscr{A}$ ($\mathscr{A} = K[x_1, \dots, x_n]$, $K = \mathbb{R}$ or $\mathbb{C}$), let $\mathscr{V}(\mathbf{I})$ denote the affine variety of $\mathbf{I}$ in $\mathbb{C}^n$. Then the Hilbert Nullstellensatz asserts that the ideal of $\mathscr{V}(\mathbf{I})$ is the radical ideal $\sqrt{\mathbf{I}}$, so that there is a bijective correspondence between radical ideals and affine varieties.

Suppose instead that one thinks of a polynomial p as the (constant coefficient) partial differential operator $p(\partial)$, i.e., as an element of $K[\partial_1, \dots, \partial_n]$ (which being naturally isomorphic to $K[x_1, \dots, x_n]$, I also

[1] A part of this work was presented at the Conference on Commutative Algebra and Algebraic Geometry, Kodaikanal, India, 1998.

[2] Current address through July 2000: Dept. of Mathematics, The University of Groningen, P.O. Box 800, 9700 Av Groningen, The Netherlands.

0196-8858/99 $30.00

denote by $\mathscr{A}$). Consider $\mathscr{D}'$ (the space of all distributions on $\mathbb{R}^n$) as an $\mathscr{A}$-module, the module structure given by differentiation. Then given any $\mathscr{A}$-submodule $\mathscr{W}$ of $\mathscr{D}'$, one can consider the $\mathscr{A}$-module morphism

$$p(\partial): \mathscr{W} \to \mathscr{W}$$
$$w \mapsto p(\partial)w$$

(I adopt the notation in Hörmander [1], except that I use $\mathscr{D}$ instead of $\mathscr{C}_0^\infty$ for the space of compactly supported, (complex-valued) smooth functions on $\mathbb{R}^n$.) Consider the kernel of $p(\partial)$ (which is an $\mathscr{A}$-submodule of $\mathscr{W}$). Denote it by $\mathscr{B}_{\mathscr{W}}(p)$, and call it, following engineering practice, the *behaviour* of p in $\mathscr{W}$. Given an ideal $\mathbf{I}$ in $\mathscr{A}$, one can then consider the $\mathscr{A}$-submodule $\bigcap_{p \in \mathbf{I}} \mathscr{B}_{\mathscr{W}}(p)$ of $\mathscr{W}$. Denote it by $\mathscr{B}_{\mathscr{W}}(\mathbf{I})$ and call it the behaviour of $\mathbf{I}$ in $\mathscr{W}$.

More generally, given an element $r(\partial) = (r_1(\partial), \dots, r_k(\partial))$ in the free module $\mathscr{A}^k$, consider the $\mathscr{A}$-module morphism

$$r(\partial): \mathscr{W}^k \to \mathscr{W}$$
$$f = (f_1, \dots, f_k) \mapsto r(\partial)f = \sum r_i(\partial)f_i.$$

Denote the kernel of the above morphism by $\mathscr{B}_{\mathscr{W}}(r)$. Given now a submodule $\mathbf{R}$ of $\mathscr{A}^k$, denote by $\mathscr{B}_{\mathscr{W}}(\mathbf{R})$ the $\mathscr{A}$-submodule $\bigcap_{r \in \mathbf{R}} \mathscr{B}_{\mathscr{W}}(r)$ of $\mathscr{W}^k$, and call it the behaviour of $\mathbf{R}$ in $\mathscr{W}^k$ or the $\mathscr{W}$-behaviour of $\mathbf{R}$. If $\mathbf{R}$ is generated by $r_1(\partial), \dots, r_l(\partial)$, then the above behaviour is just the kernel of the morphism

$$R(\partial): \mathscr{W}^k \to \mathscr{W}^l$$
$$f \mapsto (r_1(\partial)f, \dots, r_l(\partial)f). \tag{1}$$

DEFINITION. A *behaviour* in $\mathscr{W}^k$ (or a *$\mathscr{W}$-behaviour*) is an $\mathscr{A}$-submodule of the type $\mathscr{B}_{\mathscr{W}}(\mathbf{R})$ for some $\mathscr{A}$-submodule $\mathbf{R}$ of $\mathscr{A}^k$.

Aside. If $\mathscr{W}$ is also an $\mathscr{E}'$-submodule of $\mathscr{D}'$ (the $\mathscr{E}'$-module structure given by convolution, which is therefore an extension of the $\mathscr{A}$-module structure), then as $p(\partial)(u * w) = u * p(\partial)w$ for all u in $\mathscr{E}'$ and w in $\mathscr{W}$, it follows that $p(\partial)$, hence also $R(\partial)$, are $\mathscr{E}'$-module morphisms. Then a behaviour is also an $\mathscr{E}'$-submodule of $\mathscr{W}^k$. This is indeed the case for all the $\mathscr{W}$ considered in this paper.

Remark. The morphism $R(\partial)$ above can be represented by a matrix in the usual way, say,

$$R(\partial) = \begin{pmatrix} r_{11}(\partial) & \cdots & r_{1k}(\partial) \\ . & \cdots & . \\ . & \cdots & . \\ r_{l1}(\partial) & \cdots & r_{lk}(\partial) \end{pmatrix}. \tag{2}$$

Consider the image of the map given by multiplication by the transpose, $R^T\colon \mathscr{A}^l \to \mathscr{A}^k$, i.e., the submodule $\mathbf{R}$. It is an observation of Malgrange that

$$\mathrm{Hom}_{\mathscr{A}}(\mathscr{A}^k/\mathrm{Im}\, R^T, \mathscr{W}) = \mathrm{Hom}_{\mathscr{A}}(\mathscr{A}^k/\mathbf{R}, \mathscr{W}) \simeq \mathscr{B}_{\mathscr{W}}(\mathbf{R}),$$

where the isomorphism is given by

$$\phi \mapsto (\phi(\overline{e_1}), \dots, \phi(\overline{e_k}));$$

here $\overline{e_i}$ is the image of $e_i = (0, \dots, 1, \dots, 0)$ in $\mathscr{A}^k/\mathrm{Im}\, R^T$. The study of this contravariant functor $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{W})$, when $\mathscr{W}$ is either $\mathscr{C}^\infty$ or $\mathscr{D}'$, is the starting point of Oberst's fundamental paper [3]. In this paper I study this functor when $\mathscr{W}$ is the space of compactly supported functions $\mathscr{D}$ or distributions $\mathscr{E}'$, the Schwartz space $\mathscr{S}$ of rapidly decreasing functions, and the space $\mathscr{S}'$ of temperate distributions.

On the other hand, given a behaviour $\mathbf{B}$ in $\mathscr{W}^k$ (i.e., $\mathbf{B} = \mathscr{B}_{\mathscr{W}}(\mathbf{R})$ for some $\mathbf{R}$), let $\mathscr{M}(\mathbf{B})$ be the submodule of $\mathscr{A}^k$ consisting of all the elements in $\mathscr{A}^k$ that map to zero every element in $\mathbf{B}$. Clearly $\mathbf{R} \subset \mathscr{M}(\mathbf{B})$. Thus there are two assignments $\mathscr{B}_{\mathscr{W}}$ and $\mathscr{M}$ which are both inclusion reversing; i.e., $\mathbf{R}_1 \subset \mathbf{R}_2$ implies $\mathscr{B}_{\mathscr{W}}(\mathbf{R}_2) \subset \mathscr{B}_{\mathscr{W}}(\mathbf{R}_1)$ and $\mathbf{B}_1 \subset \mathbf{B}_2$ implies $\mathscr{M}(\mathbf{B}_2) \subset \mathscr{M}(\mathbf{B}_1)$. In other words, $\mathscr{B}_{\mathscr{W}}$ and $\mathscr{M}$ define a Galois connection between the partly ordered sets of submodules of $\mathscr{A}^k$ and behaviours in $\mathscr{W}^k$.

I collect some elementary consequences below.

LEMMA 1.1. *Let* $\{\mathbf{R}_i\}$ *(respectively* $\{\mathbf{B}_i\}$*) be any collection of submodules of* $\mathscr{A}^k$ *(respectively behaviours in* $\mathscr{W}^k$*). Then*

(i) $\mathscr{B}_{\mathscr{W}}(\Sigma_i \mathbf{R}_i) = \bigcap_i \mathscr{B}_{\mathscr{W}}(\mathbf{R}_i)$

(ii) $\Sigma_i \mathscr{B}_{\mathscr{W}}(\mathbf{R}_i) \subset \mathscr{B}_{\mathscr{W}}(\bigcap_i \mathbf{R}_i)$

(iii) $\mathscr{M}(\Sigma_i \mathbf{B}_i) = \bigcap_i \mathscr{M}(\mathbf{B}_i)$

(iv) $\Sigma_i \mathscr{M}(\mathbf{B}_i) \subset \mathscr{M}(\bigcap_i \mathbf{B}_i)$.

LEMMA 1.2. $\mathscr{B}_{\mathscr{W}} \circ \mathscr{M}$ *is the identity map on behaviours for any* $\mathscr{A}$*-submodule* $\mathscr{W}$*; i.e.,* $\mathscr{B}_{\mathscr{W}} \circ \mathscr{M}(\mathbf{B}) = \mathbf{B}$ *for all* $\mathscr{W}$*-behaviours* $\mathbf{B}$.

Proof. Clearly $\mathbf{B} \subset \mathscr{B}_{\mathscr{W}} \circ \mathscr{M}(\mathbf{B})$. But $\mathbf{B}$ by definition is $\mathscr{B}_{\mathscr{W}}(\mathbf{R})$ for some submodule $\mathbf{R}$ of $\mathscr{A}^k$ (for some k). Hence $\mathbf{R} \subset \mathscr{M}(\mathscr{B}_{\mathscr{W}}(\mathbf{R})) = \mathscr{M}(\mathbf{B})$. Then $\mathscr{B}_{\mathscr{W}} \circ \mathscr{M}(\mathbf{B}) \subset \mathscr{B}_{\mathscr{W}}(\mathbf{R}) = \mathbf{B}$. ∎

COROLLARY 1.1. *The correspondence* $\mathbf{B} \to \mathscr{M}(\mathbf{B})$ *(between* $\mathscr{W}$*-behaviours and submodules of free modules over* $\mathscr{A}$*) is injective for all* $\mathscr{A}$*-submodules* $\mathscr{W}$ *of* $\mathscr{D}'$.

Proof. Suppose $\mathscr{M}(\mathbf{B}_1) = \mathscr{M}(\mathbf{B}_2)$. Then $\mathbf{B}_1 = \mathscr{B}_{\mathscr{W}} \circ \mathscr{M}(\mathbf{B}_1) = \mathscr{B}_{\mathscr{W}} \circ \mathscr{M}(\mathbf{B}_2) = \mathbf{B}_2$. ∎

The correspondence $\mathbf{R} \to \mathscr{B}_{\mathscr{W}}(\mathbf{R})$ is of course not in general injective (for instance take $\mathscr{W} = 0$ (!)). Hence I make the following

DEFINITION. A submodule $\mathbf{R}$ of $\mathscr{A}^k$ is called a *Willems* submodule with respect to $\mathscr{W}$ if $\mathscr{M}(\mathscr{B}_{\mathscr{W}}(\mathbf{R})) = \mathbf{R}$.

Thus the correspondence $\mathbf{R} \to \mathscr{B}_{\mathscr{W}}(\mathbf{R})$ is bijective when restricted to the class of Willems submodules (with respect to $\mathscr{W}$). This definition is therefore analogous to the classical definition of a radical ideal.

Immediate are the following lemmata.

LEMMA 1.3. *Let $\mathbf{B} = \mathscr{B}_{\mathscr{W}}(\mathbf{R})$ be a behaviour in $\mathscr{W}^k$. Then $\mathscr{M}(\mathbf{B})$ is the smallest Willems submodule of $\mathscr{A}^k$ (with respect to $\mathscr{W}$) that contains $\mathbf{R}$.*

(This is analogous to the fact that the ideal of a variety is a radical ideal.)

LEMMA 1.4. *If $\{\mathbf{R}_i\}$ is any collection of submodules of $\mathscr{A}^k$, each Willems with respect to $\mathscr{W}$, then $\bigcap_i \mathbf{R}_i$ is also Willems with respect to $\mathscr{W}$.*

(This is analogous to the fact that an intersection of radical ideals is radical.)

LEMMA 1.5. *Let $\mathscr{W}_1 \subset \mathscr{W}_2$ be $\mathscr{A}$-submodules of $\mathscr{D}'$. Then if the submodule $\mathbf{R}$ of $\mathscr{A}^k$ is Willems with respect to $\mathscr{W}_1$ it is also Willems with respect to $\mathscr{W}_2$.*

(This corresponds to looking at varieties in extension fields.)

All this motivates the following

Problem (*The Nullstellensatz for Systems of PDE*). Determine the Willems submodules with respect to various $\mathscr{A}$-submodules of $\mathscr{D}'$.

The solution of this problem when $\mathscr{W}$ is either $\mathscr{E}^\infty$ or $\mathscr{D}'$ is due to Oberst.

THEOREM (Oberst). *Every submodule of $\mathscr{A}^k$ is Willems with respect to $\mathscr{E}^\infty$ (and hence with respect to $\mathscr{D}'$).*

This theorem is a consequence of a deep theorem of Oberst [3], which states that $\mathscr{E}^\infty$ and $\mathscr{D}'$ are injective cogenerators (i.e., that the functor $\mathrm{Hom}_{\mathscr{A}}(-, \mathscr{W})$, when $\mathscr{W}$ is either $\mathscr{E}^\infty$ or $\mathscr{D}'$, is exact, and also that $\mathrm{Hom}_{\mathscr{A}}(\mathbf{M}, \mathscr{W}) = 0$ if and only if the $\mathscr{A}$-module $\mathbf{M}$ is 0).

An immediate corollary to Oberst's theorem is the following result, which I use elsewhere in this paper.

COROLLARY 1.2. *If $\mathbf{R}$ is not Willems with respect to $\mathscr{W}$, an $\mathscr{A}$-submodule of $\mathscr{D}'$ (resp. $\mathscr{E}^\infty$), then $\mathscr{B}_{\mathscr{W}}(\mathbf{R})$ is not dense in $\mathscr{B}_{\mathscr{D}'}(\mathbf{R})$ (resp. $\mathscr{B}_{\mathscr{E}^\infty}(\mathbf{R})$).*

Proof. If $\mathbf{R}$ is not Willems with respect to $\mathscr{W}$, then $\mathbf{R}$ is strictly contained in $\mathscr{M}\mathscr{B}_{\mathscr{W}}(\mathbf{R}) = \mathbf{R}_0$, say. Hence (as $\mathscr{B}_{\mathscr{W}}(\mathbf{R}) = \mathscr{B}_{\mathscr{W}}(\mathbf{R}_0)$) it follows that $\overline{\mathscr{B}_{\mathscr{W}'}(\mathbf{R})} \subset \mathscr{B}_{\mathscr{D}'}(\mathbf{R}_0)$ (resp. $\mathscr{B}_{\mathscr{E}^\infty}(\mathbf{R}_0)$), which is, by Oberst, strictly contained in $\mathscr{B}_{\mathscr{D}'}(\mathbf{R})$ (resp. $\mathscr{B}_{\mathscr{E}^\infty}(\mathbf{R})$). ∎

In this paper I determine the Willems submodules of $\mathscr{A}^k$ with respect to $\mathscr{S}$ (I also show that a submodule is Willems with respect to $\mathscr{S}$ if and only if it is Willems with respect to $\mathscr{D}$ or $\mathscr{E}'$) and with respect to $\mathscr{S}'$. But first I explain the connection of this nullstellensatz problem with a problem in electrical engineering.

2. THE NULLSTELLENSATZ

Systems of constant coefficient PDEs are important because they occur in nature. Many of the laws of physics can be expressed as

$$\{w \mid Nw \in \operatorname{Im}(M)\}, \tag{3}$$

where N, M are systems of PDEs (assume here that the components of w are in $\mathscr{E}^\infty$ or in $\mathscr{D}'$). Such systems can always be realized as the $\mathscr{E}^\infty$ or $\mathscr{D}'$ behaviour of some submodule $\mathbf{R}$, viz. the Elimination Theorem below. Important examples include Maxwell's equations and the heat equation.

Such systems also occur in engineering. For instance, a problem of some importance in electrical engineering is to *control the vibrations of a drum* [5]. Here the law of evolution of the system is given by the wave equation

$$p(\partial)w = c^{-2}\,\frac{\partial^2 w}{\partial t^2} - \Delta w = f,$$

whose "behaviour" is captured in the kernel of

$$\begin{aligned}[][p(\partial), -1] : \mathscr{W}^2 &\to \mathscr{W}\\ (w, f) &\mapsto p(\partial)w - f\end{aligned}$$

and thus the terminology (see [6]). Relevant to this paper is the notion of *controllability* of a differential system which I now define.

Definition. A $\mathscr{D}'$ (or $\mathscr{E}^\infty$)-behaviour $\mathbf{B}$ is said to be *controllable* if for w_1 and w_2, any two elements in $\mathbf{B}$, and for U_1 and U_2 any two open subsets of $\mathbb{R}^n$ such that their closures are disjoint (i.e., $\overline{U_1} \cap \overline{U_2} = \emptyset$), there exists an element w in $\mathbf{B}$ which coincides with w_1 on U_1 and with w_2 on U_2.

This of course means that the action of w coincides with that of w_1 on test functions whose supports are contained in U_1 and with the action of w_2 on test functions whose supports are in U_2. (The support of an element in $\mathscr{W}^k$ is the union of the supports of its components.) In the case of a $\mathscr{E}^\infty$ behaviour, w is a smooth function that coincides pointwise with w_1 on U_1 and with w_2 on U_2. One says that w has *patched* up w_1 on U_1 with w_2 on U_2.

DEFINITION. Let U be an open subset of $\mathbb{R}^n$, and let V be any closed subset whose interior contains the closure of U. Let w be an element in $(\mathscr{D}')^k$. An element w_c in $(\mathscr{D}')^k$ is a *cutoff* of w with respect to U and V if w_c coincides with w on U and with 0 on V^c, the complement of V (similar definition for cutoffs of elements in $(\mathscr{E}^\infty)^k$).

The following lemma is elementary (see [4]).

LEMMA 2.1. *A $\mathscr{D}'$ (or $\mathscr{E}^\infty$) behaviour* **B** *is controllable if and only if for every element w in it and for every U and V as in the definition above, some cutoff of w with respect to U and V is also in* **B**.

Thus it follows that the compactly supported elements of a controllable behaviour are dense in it.

Motivated by the above, I also make the following *a priori* weaker definition.

DEFINITION. A $\mathscr{D}'$ (or $\mathscr{E}^\infty$) behaviour **B** is said to be *asymptotically controllable* if for every element w in it and for every U and V as in the definition above, there is a w_a in **B** which coincides with w on U and with a rapidly decreasing function on V^c.

Clearly a controllable behaviour is asymptotically controllable. Observe also that the rapidly decreasing elements of an asymptotically controllable behaviour are dense in it.

What are examples of such behaviours? A whole class is given by the following

PROPOSITION 2.1. *An exact behaviour (that is, a behaviour which is the image of some morphism $M(\partial)\colon (\mathscr{D}')^l \to (\mathscr{D}')^k$, or $M(\partial)\colon (\mathscr{E}^\infty)^l \to (\mathscr{E}^\infty)^k$) is controllable, and hence asymptotically controllable.*

Proof. Elementary; see [4]. ∎

Conversely I prove

THEOREM 2.1. *An asymptotically controllable behaviour is exact; it is therefore also controllable.*

The connection of these engineering questions with the problem of determining Willems submodules is given by the following

THEOREM 2.2. *A submodule* **R** *is Willems with respect to $\mathscr{E}'$ (or $\mathscr{D}$) if and only if $\mathscr{B}_{\mathscr{D}'}(\mathbf{R})$ (or $\mathscr{B}_{\mathscr{E}^\infty}(\mathbf{R})$) is controllable. It is Willems with respect to $\mathscr{S}$ if and only if its behaviour is asymptotically controllable.*

Thus a submodule is Willems with respect to $\mathscr{E}'$ if and only if it is so with respect to $\mathscr{D}$, if and only if it is also so with respect to $\mathscr{S}$. This implies the following density results for systems of PDEs.

COROLLARY 2.1. *Suppose that the rapidly decreasing functions in the* $\mathscr{E}^\infty$ *behaviour of a submodule* $\mathbf{R}$ *are dense in it. Then the compactly supported functions are themselves dense in it.*

Proof. If the compactly supported functions are not dense in the $\mathscr{E}^\infty$ behaviour of $\mathbf{R}$, then clearly it is not exact. By Theorem 2.1 this behaviour is not asymptotically controllable, and hence by Theorem 2.2, $\mathbf{R}$ is not Willems with respect to $\mathscr{S}$. It then follows from Corollary 1.2 that the rapidly decreasing functions cannot be dense in the $\mathscr{E}^\infty$ behaviour of $\mathbf{R}$. ∎

COROLLARY 2.2. *The compactly supported behaviour* $\mathscr{B}_{\mathscr{D}}(\mathbf{R})$, *of any submodule* $\mathbf{R}$, *is always dense in its rapidly decreasing behaviour* $\mathscr{B}_{\mathscr{S}}(\mathbf{R})$.

Proof. Let $\mathbf{R}_0$ equal $\mathscr{M}\mathscr{B}_{\mathscr{D}}(\mathbf{R}) = \mathscr{M}\mathscr{B}_{\mathscr{S}}(\mathbf{R})$. By Lemma 1.3 $\mathbf{R}_0$ is Willems with respect to $\mathscr{D}$ or $\mathscr{S}$, hence $\mathscr{B}_{\mathscr{D}}(\mathbf{R}) = \mathscr{B}_{\mathscr{D}}(\mathbf{R}_0)$ is dense in $\mathscr{B}_{\mathscr{E}^\infty}(\mathbf{R}_0)$ and hence is dense in $\mathscr{B}_{\mathscr{S}}(\mathbf{R}_0) = \mathscr{B}_{\mathscr{S}}(\mathbf{R})$. ∎

COROLLARY 2.3. *The submodule* $\mathbf{R}$ *is Willems with respect to* $\mathscr{D}$, $\mathscr{S}$ (*or* $\mathscr{E}'$) *if and only if* $\mathscr{B}_{\mathscr{D}}(\mathbf{R})$, $\mathscr{B}_{\mathscr{S}}(\mathbf{R})$ *are dense in* $\mathscr{B}_{\mathscr{E}^\infty}(\mathbf{R})$ (*or* $\mathscr{B}_{\mathscr{E}'}(\mathbf{R})$ *is dense in* $\mathscr{B}_{\mathscr{D}'}(\mathbf{R})$).

Proof. The only if part of the statement is Corollary 1.2. The if part follows from the above theorem. ∎

I also determine when a submodule is Willems with respect to $\mathscr{S}'$. To state the result, I use the following: Given a submodule $\mathbf{R}$ of $\mathscr{A}^k$, represent $\mathbf{R}$ by a $l \times k$ matrix with polynomial entries, where the l rows (as elements of $\mathscr{A}^k$) generate $\mathbf{R}$, viz. the matrix in Eq. (2). Consider the kth determinantal ideal of this matrix. Clearly this ideal depends only on the submodule $\mathbf{R}$ and not on the choice of the matrix representing it; i.e., it is independent of the choice of generators above. Denote therefore this determinantal ideal by $I_k(\mathbf{R})$. Let $\mathscr{I}\mathscr{V}(I_k(\mathbf{R}))$ be the set of purely imaginary points on the variety of $I_k(\mathbf{R})$ (i.e., $\mathscr{I}\mathscr{V}(I_k(\mathbf{R})) = \mathscr{V}(I_k(\mathbf{R})) \cap \imath\mathbb{R}^n$).

I also use the following elementary result: Given the submodule $\mathbf{R}$ of $\mathscr{A}^k$, let $\mathbf{R} = \bigcap_{i=1}^{t} \mathbf{Q}_i$ be an irredundant primary decomposition of $\mathbf{R}$ in $\mathscr{A}^k$, where $\mathbf{Q}_i$ is $\mathbf{P}_i$-primary. Suppose that $\mathbf{J}$ is an ideal such that $\mathbf{J} \subset \mathbf{P}_i$, $i = r+1, \dots, t$, for some $r < t$ and that $\mathbf{J}$ is *not* contained in the other $\mathbf{P}_i$'s. Consider the ascending chain of submodules

$$(\mathbf{R} : \mathbf{J}) \subset (\mathbf{R} : \mathbf{J}^2) \subset \cdots .$$

Then this chain stabilizes to the submodule $\bigcap_{i=1}^{r} \mathbf{Q}_i$, and this submodule is therefore independent of the primary decomposition.

In this notation the nullstellensatz with respect to $\mathscr{S}'$ is the following

THEOREM 2.3. *Let* $\mathbf{R} = \bigcap_{i=1}^{t} \mathbf{Q}_i$ *be an irredundant primary decomposition of the submodule* $\mathbf{R}$ *in* $\mathscr{A}^k$, *where* $\mathbf{Q}_i$ *is* $\mathbf{P}_i$*-primary. Suppose that the varieties of* $\mathbf{P}_1, \dots, \mathbf{P}_r$ *intersect* $\mathscr{I}\mathscr{V}(I_k(\mathbf{R}))$ *and those of* $\mathbf{P}_{r+1}, \dots, \mathbf{P}_t$ *do not. Then* $\mathscr{M}(\mathscr{B}_{\mathscr{S}'}(\mathbf{R})) = \bigcap_{i=1}^{r} \mathbf{Q}_i$, *so that* $\mathbf{R}$ *is Willems with respect to* $\mathscr{S}'$ *if and only if the variety of every* $\mathbf{P}_i$ *intersects* $\mathscr{I}\mathscr{V}(I_k(\mathbf{R}))$.

Thus these results, together with the theorem of Oberst, solve the nullstellensatz problem for systems of PDE with respect to the classical spaces.

These results depend on the following: Let $P = (p_{ij})$ be an $l \times m$ matrix with entries in $\mathscr{A}$. Consider the set $\mathbf{Q}$ of all relations of the rows of P, i.e., the set of all l-tuples $(q_1, \dots, q_l)$ in $\mathscr{A}^l$ such that

$$\sum_{i=1}^{l} q_i p_{ij} = 0, \qquad j = 1, \dots, m.$$

This set $\mathbf{Q}$ is clearly a submodule of $\mathscr{A}^l$. In this notation one has the following

THEOREM (The Fundamental Principle of Ehrenpreis–Palamodov–Oberst). *Let f be in $(\mathscr{D}')^l$. Then there exists a u in $(\mathscr{D}')^m$ such that $P(\partial)u = f$ if and only if $q(\partial)f = 0$ for all q in* $\mathbf{Q}$. (*If f is smooth then there is a smooth u as above.*)

Remark. The fundamental principle in the $\mathscr{E}^\infty$-category (due to Ehrenpreis and Palamodov) is classical (see, for instance, Theorem 7.6.13 in [2]). This theorem for $\mathscr{D}'$ is due to Oberst [3] and follows from the fact that $\mathscr{D}'$ is an injective cogenerator (quoted in the preceding section).

COROLLARY 2.4 (The Elimination Theorem). *The $\mathscr{A}$-submodule given by Eq.* (3) *is the kernel of some system as in Eq.* (1).

Proof. Given the submodule defined by N and M as in (3), let $\mathbf{Q}$ be the module of relations of the rows of M (i.e., let P above equal M). Let $\mathbf{Q}$ be generated by g elements, say $(\mathbf{q}_{11}, \dots, \mathbf{q}_{1l}), \dots, (\mathbf{q}_{g1}, \dots, \mathbf{q}_{gl})$. This choice of generators defines a morphism $Q(\partial)$: $(\mathscr{D}')^l \to (\mathscr{D}')^g$ whose matrix representation is (q_{ij}). By the Fundamental Principle $Mu = f$ if and only if $Qf = 0$. Consider now the morphism defined by the matrix QN. Suppose w is in the kernel of QN; i.e., suppose that Nw is in the kernel of Q. This implies that Nw is in the image of M and thus that w is in (3). Conversely if $Nw = Mu$ for some u, then $QNw = QMu = 0$. ∎

3. PROOFS

I now prove the statements listed above. I first show that the $\mathscr{C}^\infty$ or $\mathscr{D}'$ behaviour of $\mathbf{R}$ is exact $\Leftrightarrow$ it is controllable $\Leftrightarrow$ it is asymptotically controllable $\Leftrightarrow$ $\mathbf{R}$ is Willems with respect to $\mathscr{D}$, $\mathscr{E}'$, or $\mathscr{S}$.

The following proposition characterizes behaviours that are exact.

PROPOSITION 3.1. *A $\mathscr{D}'$ or $\mathscr{C}^\infty$ behaviour $\mathbf{B}$ given by a submodule $\mathbf{R}$ of $\mathscr{A}^k$ is exact if and only if $\mathscr{A}^k/\mathbf{R}$ is torsion free.*

Proof. Given the behaviour $\mathbf{B}$, i.e., the kernel of $R(\partial)$: $(\mathscr{D}')^k \to (\mathscr{D}')^l$ (where R is a matrix whose l rows generate $\mathbf{R}$), consider subbehaviours of $\mathbf{B}$ which are exact. Thus let the image of $M(\partial)$: $(\mathscr{D}')^{g'} \to (\mathscr{D}')^k$ be contained in the kernel of R. It follows then that

$$\sum_{j=1}^{k} r_{ij} m_{jh} = 0, \qquad i = 1, \dots, l, h = 1, \dots, g' \tag{4}$$

(where the r_{ij} and m_{ij} are the entries of R and M, respectively). This means that the columns of M are relations between the columns of R. Conversely, relations between columns of R determine a morphism whose image is a subbehaviour of $\mathbf{B}$.

Consider the module $\mathbf{M}_0$ of all relations between the columns of R. Generators of this module, say g in number, determine a morphism, $M_0(\partial)$: $(\mathscr{D}')^g \to (\mathscr{D}')^k$, whose image is clearly the largest subbehaviour of $\mathbf{B}$ which is exact.

Consider next the module of relations $\mathbf{R}_0$ of the rows of the matrix representing M_0. By (4) it follows that $\mathbf{R} \subset \mathbf{R}_0$. By the Fundamental Principle the image of M_0 is precisely the kernel of R_0 (which is the morphism determined by any set of generators of $\mathbf{R}_0$ as above). Thus the behaviour of $\mathbf{R}$ is exact if and only if $\mathbf{R} = \mathbf{R}_0$.

Let $\mathscr{K}$ be the quotient field of $\mathscr{A}$. After tensoring by $\mathscr{K}$, the submodules $\mathscr{K} \otimes \mathbf{R}$, $\mathscr{K} \otimes \mathbf{M}_0$, and $\mathscr{K} \otimes \mathbf{R}_0$ are vector spaces. Thus if the rank of $\mathscr{K} \otimes \mathbf{R}$ is i, then the rank of $\mathscr{K} \otimes \mathbf{M}_0$ is $k - i$, which in turn implies that the rank of $\mathscr{K} \otimes \mathbf{R}_0$ is again i. As $\mathscr{K} \otimes \mathbf{R}$ is contained in $\mathscr{K} \otimes \mathbf{R}_0$ ($\mathscr{K}$ is $\mathscr{A}$-flat (!)) it follows that $\mathscr{K} \otimes \mathbf{R} = \mathscr{K} \otimes \mathbf{R}_0$.

So if r_0 is in $\mathbf{R}_0$, then $1 \otimes r_0$ in $\mathscr{K} \otimes \mathbf{R}_0$ is also in $\mathscr{K} \otimes \mathbf{R}$. Thus

$$1 \otimes r_0 = \frac{p_1}{q_1} \otimes r_1 + \cdots + \frac{p_t}{q_t} \otimes r_t$$

for some p_i, q_i in $\mathscr{A}$ and r_i in $\mathbf{R}$. Let q be the product of the q_i. Then

$$1 \otimes q r_0 = q_1' p_1 \otimes r_1 + \cdots + q_t' p_t \otimes r_t,$$

where $q_i' = q/q_i$. This implies that $1 \otimes q r_0$ is equal to $1 \otimes r$ for some r in $\mathbf{R}$. Thus $1 \otimes (q r_0 - r) = 0$, which implies that $r = q r_0$. Hence for any r_0 in

$\mathbf{R}_0$, there is some nonzero element q in $\mathscr{A}$ such that qr_0 is in $\mathbf{R}$. Similarly it follows that if, for some r in $\mathscr{A}^k$, qr is in $\mathbf{R}$ for some nonzero q in $\mathscr{A}$, then r is in $\mathbf{R}_0$.

Thus $\mathbf{R}_0/\mathbf{R}$ is the set of torsion elements in $\mathscr{A}^k/\mathbf{R}$, which is to say that $\mathbf{R} = \mathbf{R}_0$ if and only if $\mathscr{A}^k/\mathbf{R}$ is torsion free. ∎

Proof of Theorem 2.1. Suppose that the behaviour of $\mathbf{R}$ is not the image of any morphism. Then by the above proposition $\mathscr{A}^k/\mathbf{R}$ is not torsion free. So let p be an element in $\mathscr{A}^k$ not in $\mathbf{R}$ but such that ap is in $\mathbf{R}$ for some nonzero a in $\mathscr{A}$.

Consider the morphisms

$$P: \mathscr{B}_{\mathscr{D}'}(\mathbf{R}) \to \mathscr{D}'$$
$$w \mapsto p(\partial)w$$

and

$$A: \mathscr{D}' \to \mathscr{D}'$$
$$f \mapsto a(\partial)f$$

with p and a as above. By the theorem of Oberst, P is not the zero morphism as p is not in $\mathbf{R}$. However, as ap is in $\mathbf{R}$, the composition $A \circ P$: $\mathscr{B}_{\mathscr{D}'}(\mathbf{R}) \to \mathscr{D}'$ is the zero morphism.

Let now w be any element in $\mathscr{B}_{\mathscr{D}'}(\mathbf{R})$ which is not in the kernel of P. Let U be any bounded open subset of $\mathbb{R}^n$ where $P(\partial)w$ is not identically zero, and let V be any compact set whose interior contains the closure of U. Let w_a be any element of $(\mathscr{D}')^k$ which coincides with w on U and with a rapidly decreasing function (i.e., an element of $\mathscr{S}$) on V^c. Then $P(\partial)w_a$ is nonzero. If the behaviour of $\mathbf{R}$ were asymptotically controllable, then some element such as w_a, say w_1, must be in it. This implies that $P(\partial)w_1$ (equal to u say) must be in the kernel of A. This u also coincides with an element of $\mathscr{S}$, say f, on V^c (as $\mathscr{S}$ is stable under differentiation).

Now consider $f - u$ which has compact support. Then $a(\partial)(f-u) = a(\partial)f$. Taking Fourier transforms, $a(\iota x)\widehat{(f-u)}(x) = a(\iota x)\hat{f}(x)$. By Paley–Wiener, $\widehat{(f-u)}(x)$ is an analytic function. This implies that $\widehat{(f-u)}(x) = \hat{f}(x)$, which in turn implies that u is identically zero. This contradiction proves the theorem. ∎

Proof of Theorem 2.2. I prove that $\mathbf{R}$ is Willems with respect to $\mathscr{S}$; i.e., $\mathbf{R} = \mathscr{M}(\mathscr{B}_{\mathscr{S}}(\mathbf{R}))$, if and only if the behaviour of $\mathbf{R}$ is asymptotically controllable. (The proofs of the other statements in the theorem are similar (see also [4]).)

If $\mathbf{R}$ is not Willems with respect to $\mathscr{S}$, then by Corollary 1.2, the rapidly decreasing functions cannot be dense in the behaviour of $\mathbf{R}$. As observed earlier, this implies that the behaviour cannot be asymptotically controllable.

Conversely, suppose that the behaviour of $\mathbf{R}$ is not asymptotically controllable, so that by the above theorem it is not an image. Hence, in the notation of Proposition 3.1, $\mathbf{R}$ is strictly contained in $\mathbf{R}_0$. This implies that $\mathscr{B}_{\mathscr{S}}(\mathbf{R}_0) \subset \mathscr{B}_{\mathscr{S}}(\mathbf{R})$. In fact, I claim that these two behaviours are identical, i.e., that every p in $\mathbf{R}_0$ maps to zero every element in $\mathscr{B}_{\mathscr{S}}(\mathbf{R})$. For suppose that this were not true for some p in $\mathbf{R}_0$. By Proposition 3.1 there is a nonzero a in $\mathscr{A}$ such that ap is in $\mathbf{R}$. Thus ap maps to zero every element in $\mathscr{B}_{\mathscr{S}}(\mathbf{R})$. This would lead to an absurdity just as in the proof of Theorem 2.1 above. Hence $\mathbf{R}$ is strictly contained in $\mathbf{R}_0 \subset \mathscr{M}(\mathscr{B}_{\mathscr{S}}(\mathbf{R}_0)) = \mathscr{M}(\mathscr{B}_{\mathscr{S}}(\mathbf{R}))$. Thus $\mathbf{R}$ is not Willems with respect to $\mathscr{S}$. ∎

Proof of Theorem 2.3. Let $\mathbf{R}_0$ denote $\bigcap_{i=1}^{r} \mathbf{Q}_i$. By the discussion preceding the statement of Theorem 2.3 in the previous section, it follows that $\mathbf{R}_0$ is independent of the primary decomposition of $\mathbf{R}$ (take the ideal $\mathbf{J}$ there to be $\bigcap_{i=r+1}^{t} \mathbf{P}_i$). I first show that the $\mathscr{S}'$ behaviour of $\mathbf{R}_0$ equals that of $\mathbf{R}$. As $\mathbf{R} \subset \mathbf{R}_0$, it suffices to show that $\mathscr{B}_{\mathscr{S}'}(\mathbf{R}) \subset \mathscr{B}_{\mathscr{S}'}(\mathbf{R}_0)$. If this is not true, then there is some f in $\mathscr{B}_{\mathscr{S}'}(\mathbf{R})$ and some p in $\mathbf{R}_0 \backslash \mathbf{R}$ such that $p(\partial)f \neq 0$. However, for every a in the ideal $(\mathbf{R} : p)$, $a(\partial)(p(\partial)f) = 0$. Taking Fourier transforms then implies that for every such a, $a(\iota x)(\widehat{p(\partial)f})(x) = 0$, and thus that $\iota(\operatorname{supp}(\widehat{p(\partial)f}))$ is contained in $\mathscr{V}(a) \cap \iota\mathbb{R}^n$. As $\mathbf{R} = \bigcap_{i=1}^{t} \mathbf{Q}_i$, $(\mathbf{R} : p) = \bigcap_{i=1}^{t}(\mathbf{Q}_i : p)$, and as p is in $\mathbf{R}_0 \backslash \mathbf{R}$—i.e., as p is in every one of $\mathbf{Q}_1, \dots, \mathbf{Q}_r$ and not in at least one of the other $\mathbf{Q}_i$'s—it follows that $\sqrt{(\mathbf{R} : p)}$ is equal to the intersection of a subset of $\mathbf{P}_{r+1}, \dots, \mathbf{P}_t$ ($\mathbf{Q}_i$ is $\mathbf{P}_i$-primary). Thus $\mathscr{V}(\mathbf{R} : p)$ is the union of some of the $\mathscr{V}(\mathbf{P}_{r+1}), \dots, \mathscr{V}(\mathbf{P}_t)$, from which it follows that $\iota(\operatorname{supp}(\widehat{p(\partial)f}))$ is contained in $\bigcup_{i=r+1}^{t} \mathscr{I}\mathscr{V}(\mathbf{P}_i)$.

Multiplying a matrix representation of $\mathbf{R}$ (as a $l \times k$ matrix, say) on the left by the adjoints of its various $k \times k$ submatrices (each suitably embedded in a $k \times l$ matrix, the other entries being 0), one obtains $k \times k$ diagonal matrices with entry an element of the determinantal ideal $I_k(\mathbf{R})$. In fact, a diagonal matrix with entry any element of $I_k(\mathbf{R})$ can be obtained by multiplying this matrix representation of $\mathbf{R}$ by a suitable $\mathscr{A}$-linear combination of these adjoints. From this it follows that for every $g = (g_1, \dots, g_k)$ in $\mathscr{B}_{\mathscr{S}'}(\mathbf{R})$, and for every q in $I_k(\mathbf{R})$, $q(\partial)g_i = 0$. In particular, every such $q(\partial)$ maps the components of the f chosen above to 0. Hence also $q(\partial)(p(\partial)f) = 0$, which implies by Fourier transformation that $\iota(\operatorname{supp}(\widehat{p(\partial)f}))$ is also contained in $\mathscr{I}\mathscr{V}(I_k(\mathbf{R}))$. But by assumption $\bigcup_{i=r+1}^{t} \mathscr{I}\mathscr{V}(\mathbf{P}_i) \cap \mathscr{I}\mathscr{V}(I_k(\mathbf{R})) = \emptyset$. This implies that $p(\partial)f$ equals 0, in contradiction to the assumption above.

I now show that $\mathbf{R}_0$ is the largest submodule of $\mathscr{A}^k$ with the same $\mathscr{S}'$-behaviour as that of $\mathbf{R}$. So let $p = (p_1, \ldots, p_k)$ be any element of $\mathscr{A}^k \setminus \mathbf{R}_0$, and consider the exact sequence

$$0 \to \mathscr{A}/(\mathbf{R}_0 : p) \xrightarrow{p} \mathscr{A}^k/\mathbf{R}_0 \xrightarrow{\pi} \mathscr{A}^k/\mathbf{R}_0 + (p) \to 0,$$

where the morphism p above maps the class of a to the class of ap, and π is the canonical surjection. By Oberst, $\mathscr{C}^\infty$ is an injective cogenerator, hence it follows that the sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_{\mathscr{A}}\big(\mathscr{A}^k/\mathbf{R}_0 + (p), \mathscr{C}^\infty\big) \to \operatorname{Hom}_{\mathscr{A}}\big(\mathscr{A}^k/\mathbf{R}_0, \mathscr{C}^\infty\big) \\ &\xrightarrow{p(\partial)} \operatorname{Hom}_{\mathscr{A}}\big(\mathscr{A}/(\mathbf{R}_0 : p), \mathscr{C}^\infty\big) \to 0 \end{aligned}$$

is exact (where the morphism $p(\partial)$ above maps $f = (f_1, \ldots, f_k)$ to $\Sigma p_i(\partial) f_i$) and also that $\operatorname{Hom}_{\mathscr{A}}(\mathscr{A}/(\mathbf{R}_0 : p))$ is nonzero (as $(\mathbf{R}_0 : p)$ is not all of $\mathscr{A}$). Observe that the above sequence is, by Malgrange, $0 \to \mathscr{B}_{\mathscr{C}^\infty}(\mathbf{R}_0 + (p)) \to \mathscr{B}_{\mathscr{C}^\infty}(\mathbf{R}_0) \xrightarrow{p(\partial)} \mathscr{B}_{\mathscr{C}^\infty}((\mathbf{R}_0 : p)) \to 0$.

As $\mathbf{R}_0 = \bigcap_{i=1}^r \mathbf{Q}_i$, p is not in at least one of these $\mathbf{Q}_i$, so that $\mathscr{V}(\mathbf{R}_0 : p)$ is the union of some of the $\mathscr{V}(\mathbf{P}_1), \ldots, \mathscr{V}(\mathbf{P}_r)$. But by assumption each of these varieties intersects $\mathscr{I}\mathscr{V}(I_k(\mathbf{R}))$, and hence so does the variety of $(\mathbf{R}_0 : p)$. Let m be some point in this intersection. Consider the smooth function $e^{\langle x, m \rangle}$ in $\operatorname{Hom}_{\mathscr{A}}(\mathscr{A}/(\mathbf{R}_0 : p), \mathscr{C}^\infty)$, i.e., in the $\mathscr{C}^\infty$-behaviour of the ideal $(\mathbf{R}_0 : p)$. As m has purely imaginary coordinates, $e^{\langle x, m \rangle}$ is actually a temperate distribution. Observe now that $e^{\langle x, m \rangle}$ is the image of an element in the $\mathscr{S}'$-behaviour of $\mathbf{R}_0$, i.e., of an element in $\operatorname{Hom}_{\mathscr{A}}(\mathscr{A}^k/\mathbf{R}_0, \mathscr{S}')$. (This is because the closed linear hull of the set of elements of the form $u(x)e^{\langle x, - \rangle}$, where the components of u are polynomials, are dense in the $\mathscr{C}^\infty$-behaviour of any submodule (viz. Theorem 7.6.14 in [2]).) By exactness of the sequence above, it then follows that this element in the $\mathscr{S}'$-behaviour of $\mathbf{R}_0$ cannot be in the $\mathscr{S}'$-behaviour of $\mathbf{R}_0 + (p)$. This proves that $\mathbf{R}_0$ is Willems with respect to $\mathscr{S}'$. ∎

4. COMPLEXES OF PARTIAL DIFFERENTIAL OPERATORS

I now consider, briefly, some questions on complexes of partial differential operators suggested by the above results.

Let $\mathbf{R}$ be a submodule of $\mathscr{A}^k$, generated by l elements, say, and consider the $\mathscr{E}'$-module morphism R determined by it,

$$R(\partial) : (\mathscr{D}')^k \to (\mathscr{D}')^l. \tag{5}$$

Consider a complex in which (5) is embedded:

$$\cdots \to (\mathscr{D}')^{k_i} \xrightarrow{R_{-i}} \cdots \xrightarrow{R_{-2}} (\mathscr{D}')^{k_1} \xrightarrow{R_{-1}} (\mathscr{D}')^{k} \xrightarrow{R} (\mathscr{D}')^{l} \xrightarrow{R_1} (\mathscr{D}')^{l_1} \xrightarrow{R_2} \cdots \xrightarrow{R_i} (\mathscr{D}')^{l_i} \to \cdots.$$

I consider the following questions:

Given the system (5), when is it possible to embed it in a complex (i.e., to construct a complex such as above) which is exact? If this is not possible, what is the complex with minimal cohomology in which the system can be embedded? Is this complex of finite length?

Let R also denote the $(l \times k)$ matrix defined by the morphism R. Inductively construct $\{R_i,\ i \geq 0\}$ as follows. Let R_i equal to R for $i = 0$. Given the $(l_i \times l_{i-1})$ matrix R_i, let $\mathbf{R}_{i+1}$ be the submodule of A^{l_i} generated by the relations of the rows of R_i. Suppose it is generated by l_{i+1} elements. Represent $\mathbf{R}_{i+1}$ by the corresponding $(l_{i+1} \times l_i)$ matrix R_{i+1}.

From this recursive definition it is clear that, for every i, the following sequence is short exact

$$0 \to \mathbf{R}_{i+1} \to \mathscr{A}^{l_i} \xrightarrow{\pi_i} \mathbf{R}_i \to 0, \tag{6}$$

where π_i maps a basis of $\mathscr{A}^{l_i}$ to the generators of $\mathbf{R}_i$ (chosen in the inductive construction above). Patching up these short exact sequences one gets the following free resolution of $\mathbf{R}$:

$$\cdots \xrightarrow{\pi_{i+1}} \mathscr{A}^{l_i} \xrightarrow{\pi_i} \mathscr{A}^{l_{i-1}} \to \cdots \xrightarrow{\pi_1} \mathscr{A}^{l} \xrightarrow{\pi_0} \mathbf{R} \to 0.$$

As the global dimension of $\mathscr{A}$ is n, it follows that $\operatorname{Im} \pi_n$ is a projective module, hence free. Thus the following is a free (finite length) resolution of $\mathbf{R}$:

$$0 \to \operatorname{Im} \pi_n \to \mathscr{A}^{l_{n-1}} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_0} \mathbf{R} \to 0.$$

Chopping up this resolution into short exact sequences yields the sequences (6) for $i = 1, \ldots, n-1$ and the fact that $\mathbf{R}_n$ (being equal to $\operatorname{Im} \pi_n$) is a free $\mathscr{A}$-module. But to say that that $\mathbf{R}_n$ is free is to say that $\mathbf{R}_{n+1}$ is 0. By Ehrenpreis–Palamodov–Oberst this implies that the morphism $R_n(\partial)$: $(\mathscr{D}')^{l_{n-1}} \to (\mathscr{D}')^{l_n}$ is surjective. Also the image of R_i equals the kernel of R_{i+1}, $i = 1, \ldots, n-1$ (and the image of R equals the kernel of R_1). Thus the system (5) can be extended to the right by the *finite length exact* complex

$$(\mathscr{D}')^{k} \xrightarrow{R} (\mathscr{D}')^{l} \xrightarrow{R_1} (\mathscr{D}')^{l_1} \xrightarrow{R_2} \cdots \to (\mathscr{D}')^{l_{n-1}} \xrightarrow{R_n} (\mathscr{D}')^{l_n} \to 0.$$

Consider now the construction of the left part of the complex. The problem is to first determine a morphism $R_{-1}(\partial)$: $(\mathscr{D}')^{k'} \to (\mathscr{D}')^k$ such that $\operatorname{Im} R_{-1} \subset \operatorname{Ker} R$ and such that the 0th cohomology, i.e., $\operatorname{Ker} R/\operatorname{Im} R_{-1}$, is minimal. It is clear from the proof of Proposition 3.1 that R_{-1} must therefore be chosen to be the morphism M_0 (in the notation there). Then the 0th cohomology is indeed minimal and vanishes if and only if $\mathscr{A}^k/\mathbf{R}$ is torsion-free.

Consider now the morphism M_0. As any nonzero $p(\partial)$ is surjective on $\mathscr{D}'$ (the Fundamental Principle), it follows that the columns of the matrix representation of M_0 can be assumed to be $\mathscr{A}$-independent. For if this is not so, i.e., if only some k_1 columns of M_0 are $\mathscr{A}$-independent, then deleting the other columns yields a morphism, say $R_{-1}(\partial)$: $(\mathscr{D}')^{k_1} \to (\mathscr{D}')^k$, whose image equals the image of M_0. Thus the 0th cohomology of the complex remains unchanged. But now as the columns of R_{-1} are $\mathscr{A}$-independent, it follows by Ehrenpreis–Palamodov–Oberst that there is no nonzero morphism whose image is contained in the kernel of R_{-1}. Thus one way to prolong the complex to the left is by

$$0 \to (\mathscr{D}')^{k_1} \xrightarrow{R_{-1}} (\mathscr{D}')^k \xrightarrow{R} (\mathscr{D}')^l.$$

(This is equivalent to the fact that there are no compactly supported elements in the kernel of R_{-1}). Hence the -1st cohomology of this complex is $\operatorname{Ker} R_{-1}$. Furthermore it can be shown that this cohomology is a finite $\mathscr{E}'$-module (in fact a finite-dimensional $\mathbb{R}$-vector space) if and only if $\mathscr{A}/I_{k_1}(R_{-1})$ is Artinian, where $I_{k_1}(R_{-1})$ is the k_1th determinantal ideal of R_{-1}; see [4].

Thus finally it follows that the system (5) can be embedded in a complex with nonvanishing cohomology at most at levels 0 and -1. A more detailed study of this complex and its cohomology will be reported elsewhere.

ACKNOWLEDGMENT

I thank J. K. Verma for many useful conversations.

Note Added in Proof. It is possible to strengthen Theorem 2.3 to the following—

THEOREM. *Let $\mathbf{R} = \cap_{i=1}^{t} \mathbf{Q}_i$ be an irredundant primary decomposition of the submodule $\mathbf{R}$ in $\mathscr{A}^k$, where $\mathbf{Q}_i$ is $\mathbf{P}_i$-primary. Suppose that the affine varieties (in $\mathbb{C}^n$) of $\mathbf{P}_1, \dots, \mathbf{P}_r$ contain purely imaginary points (i.e., intersect $\iota\mathbb{R}^n$) and those of $\mathbf{P}_{r+1}, \dots, \mathbf{P}_t$ do not. Then $\mathscr{M}(\mathscr{B}_{\mathscr{S}'}(\mathbf{R})) = \cap_{i=1}^{r} \mathbf{Q}_i$, so that $\mathbf{R}$ is Willems with respect to $\mathscr{S}'$ if and only if the variety of every associated prime of $\mathscr{A}^k/\mathbf{R}$ contains purely imaginary points.*

Details appear in S. Shankar, "The Lattice Structure of Behaviours," preprint, Dept. of Electrical Engg., IIT, Bombay, 1999.

REFERENCES

1. L. Hörmander, "The Analysis of Linear Partial Differential Operators, I and II," Grundlehren der Mathematischen Wissenschaften 256 and 257, Springer-Verlag, Berlin/New York, 1983.
2. L. Hörmander, "An Introduction to Complex Analysis in Several Variables," 3rd ed., North-Holland, Amsterdam, 1990.
3. U. Oberst, Multidimensional constant linear systems, *Acta Appl. Math.* **20** (1990), 1–175.
4. H. Pillai and S. Shankar, A behavioural approach to control of distributed systems, *SIAM J. Control Optim.* **37** (1999), 388–408.
5. S. Shankar, Can one control the vibrations of a drum? *Multidimensional Syst. Signal Processing*, special issue (Jan 2000), to appear.
6. J. C. Willems, Paradigms and puzzles in the theory of dynamical systems, *IEEE Trans. Automat. Contr.*, **36** (1991), 259–294.